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ITERATIVE SOLUTION OF LINEAR FUNCTIONAL EQUATIONS IN BANACH SPACES

A THESIS

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Robert Ernest Stanford

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Approved:

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Chairman

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CHAPTER I

INTRODUCTION

This thesis is concerned with iterative solution of the equation

$$u - Tu = f, \qquad (1)$$

where f is a given element in a Banach space X, and T is a given bounded linear operator mapping X into X. The iterative scheme considered is specified by

$$x_{n+1} = Tx_n + f \qquad (n=0,1,2,\cdots) \qquad (2)$$

with x_o an arbitrarily chosen element of X.

Both the equation and the iterative scheme defined have been studied extensively in recent years, as for example by H. Bialy, M. A. Krasnaselskii, W. V. Petryshyn, and F. E. Browder. The aim in this thesis is to supply a rather detailed account of the chief theorems in a recent paper by F. E. Browder and W. V. Petryshyn, *The solution by iteration of linear functional equations in Banach spaces*, Bulletin of the American Mathematical Society, Volume 72 (1966), 566-570. This paper contains some of the strongest results known on the problem concerned. Proofs in the above paper are in brief form, and some proofs are sketched or outlined only. It is hoped that the present detailed treatment will be of interest to those wishing to become acquainted with the known results, possibly with a view to new applications as to extension of the results.

To give an idea of the type of theorems concerned, we state two of the theorems here.

I. Let X be a Banach space, and let T be a bounded linear operator on X which is asymptotically convergent. Then:

(a) If f is an element in the range of the operator I - T (where I is the identity operator on X), then the sequence $\{x_n\}$ defined in (2) for any initial approximation x_o will converge to a solution of Equation (1).

(b) If any subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges to an element y of X, then y is a solution of (1), and the whole sequence $\{x_n\}$ converges to y.

(c) If X is reflexive and the sequence $\{x_n\}$ is bounded, then $\{x_n\}$ converges to a solution of (1).

II. Let T be a self-adjoint non-expanding operator on a Hilbert space H. Then T is asymptotically convergent if and only if -1 is not an eigenvalue of T.

A considerable amount of background material on Banach spaces, Hilbert spaces, and bounded linear functionals and operators, is included. The aim is to make the reading as easy as possible, and to show how various fundamental theorems in the theory of normed linear spaces are used in the proofs of results related to the iteration scheme. Some lengthy proofs are omitted, as well as proofs for some results commonly discussed in basic graduate analysis courses. In every case, an appropriate reference is given for a proof not included. Chapter II consists largely of background material on Banach spaces. Chapter III contains the fundamental theorems in the Browder-Petryshyn paper cited above. Chapter IV concerns the specialization of the results of Chapter III to Hilbert spaces, and includes some general remarks on Hilbert spaces which are pertinent.

CHAPTER II

PRELIMINARY TOPICS IN LINEAR ANALYSIS

2-1 In the following, X and Y will represent normed linear spaces, and $\|x\|$ will represent the norm of an element $x \in X$.

A sequence of elements $\{x_n\}$ in X is said to *converge in norm* to an element $x \in X$ if for every $\varepsilon > 0$, there exists an integer $N \geq 1$ such that

$$\|x_n - x\| < \varepsilon \quad \text{whenever} \quad n \geq N.$$

This is usually written $\|x_n - x\| \to 0$ as $n \to \infty$. Convergence in the norm of X is often called *strong convergence* in X (as in Lorch [6]) for reasons that will become clear later.

The sequence $\{x_n\}$ in X is a *Cauchy-sequence* in the norm of X if, for every $\varepsilon > 0$, there exists an integer $N \geq 1$ such that

$$\|x_n - x_m\| < \varepsilon \quad \text{whenever } n \geq N \text{ and } m \geq N.$$

Every sequence of elements in X which converges in norm is a Cauchy sequence in norm, but the converse is not true in general. If $\{x_n\}$ is a Cauchy sequence in the norm of X, there need be no element x in X for which $\|x_n - x\| \to 0$ as $n \to \infty$. A normed linear space X for which every Cauchy sequence of elements in X converges in norm to an element of X is said to have the property of *completeness*, or to be *complete;* a complete normed linear space is called a *Banach space*.

2-2 If X and Y are normed linear spaces, a mapping T from X into Y is a *linear transformation* if

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

for every x and y in X, and for any scalars α and β.

The *norm* of a linear transformation $T : X \to Y$, denoted by $\|T\|$, is defined by

$$\|T\| = \sup\{\ \|Tx\| : x \in X,\ \|x\| \leq 1\}$$

Note that for the element x we apply the norm associated with X, and for Tx the norm associated with Y. It can be proved that if X consists of more than the zero element, then

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in X,\quad x \neq 0\right\}$$

and

$$\|T\| = \sup\{\ \|Tx\| : x \in X,\quad \|x\| = 1\}$$

If $\|T\| < \infty$, T is called a *bounded linear transformation.* In this case, it can be shown that

$$\|Tx\| \leq \|T\| \; \|x\| \quad \text{for every} \quad x \in X$$

A linear transformation T is *continuous* at $x_o \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|Tx - Tx_o\| < \varepsilon \quad \text{whenever} \; \|x - x_o\| < \delta.$$

The transformation T is *continuous on X* if it is continuous at every $x \in X$. A linear transformation is continuous on X if and only if it is bounded. For a proof, see Rudin [8].

2-3 The set of all bounded linear transformations from X into Y is denoted by L(X,Y). A transformation from a space X into itself is called an *operator on X,* and we refer to L(X,X) as the set of all bounded linear operators on X. A transformation from X into the real or complex field (each denoted temporarily by K) is a *functional.* Analogously, L(X,K) is the set of all bounded linear functionals on X. In this case the field K is regarded as a one dimensional linear space over K, and the absolute value is used as the norm on K. L(X,K) is called the *dual* (or *conjugate*) space of X, and is usually denoted by X^* (or occasionally by X').

Let $\{T_n\}$ be a sequence of transformations in L(X,Y). We say that $\{T_n\}$ *converges uniformly* to a transformation $T \varepsilon L(X,Y)$ if the sequence $\{T_n\}$ converges to T in the norm of L(X,Y), that is, if

$$\|T_n - T\| \to 0 \quad \text{as} \quad n \to \infty.$$

The sequence $\{T_n\}$ *converges strongly* to a transformation $T \varepsilon L(X,Y)$ if

$$\|T_n x - Tx\| \to 0 \quad \text{as} \quad n \to \infty$$

for *each* x in X. Note that in the latter case the norm is that of the space Y. It should be pointed out here that usage of the word "strong" is not completely consistent. We had earlier given the expression "strong convergence" in X as an alternative term for "convergence in the norm" of X. However, when we are dealing with the special space L(X,Y), even though it is itself a normed linear space, strong convergence will be understood to have the meaning given in this section. This terminology is standard.

2-4 Banach Steinhaus Theorem. Suppose X is a Banach space, and Y a normed linear space. Let $\{T_\alpha : \alpha \varepsilon I\}$ (where I is an indexing set, possibly uncountable) be a nonempty collection of bounded linear transformations in L(X,Y). Then either

(1) There exists a positive real number M such that

$$\|T_\alpha\| \leq M \quad \text{for every} \quad \alpha \varepsilon I,$$

or

(2) $\sup \{ \|T_\alpha x\| : \alpha \varepsilon I\} = \infty$ for all x in some dense G_δ set in X. (A G_δ set is a set which is a countable intersection of open sets.)

For a proof of this important theorem, see Rudin [8]. A useful consequence is the following result:

2-5 *Corollary.* Suppose $\{T_n\}$ is a sequence in $L(X,Y)$, where X is a Banach space, and suppose

$$\|T_n x - Tx\| \to 0 \quad \text{as} \quad n \to \infty$$

for every $x \in X$. Then $T \in L(X,Y)$, and

$$\|T\| \le \liminf \{ \|T_n\| \} .$$

Proof: Fix x. For every $\varepsilon > 0$, there exists an integer $N \ge 1$ such that

$$\|T_n x - Tx\| < \varepsilon \quad \text{whenever} \quad n \ge N.$$

Then

$$\|T_n x\| \le \|Tx\| + \varepsilon \quad \text{for} \quad n \ge N.$$

Therefore, for each $n \ge 1$, we have

$$\|T_n x\| \le \max \{ \|Tx\| + \varepsilon, \ \|Tx\|, \ \|T_2 x\|, \ \dots, \ \|T_N x\| \}.$$

Consequently

$$S_x = \sup \{ \|T_n x\| : n \ge 1 \} < \infty.$$

This inequality holds for *each* x in X, and therefore the number S_x cannot be infinite for all x in some dense G_δ set in X. Thus the Banach-Steinhaus theorem establishes the existence of a real number $M > 0$ such that

$$\|T_n\| \leq M \quad \text{for all} \quad n \geq 1.$$

Now since

$$\Big| \|T_n x\| - \|Tx\| \Big| \leq \|T_n x - Tx\| \; ,$$

we have that

$$\lim_{n\to\infty} \|T_n x\| = \|Tx\| \quad \text{for each} \quad x \in X.$$

Since

$$\|T_n x\| \leq \|T_n\| \; \|x\| \quad \text{for each} \quad n \geq 1,$$

it follows that when $\|x\| \leq 1$,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \lim_{n\to\infty} \inf\{ \|T_n\| \; \|x\| \} \leq \lim_{n\to\infty} \inf \{ \|T_n\| \}.$$

Then

$$\|T\| = \sup\{ \|Tx\| : \|x\| \leq 1\} \leq \lim_{n\to\infty} \inf \{ \|T_n\| \} \leq M < \infty.$$

With the help of this corollary, we can establish the following theorem.

2-6 *Theorem.* If X is a normed linear space, then the dual space X^* is a Banach space.

Proof. The linearity of X^* is obvious. To verify completeness in the norm of X^*, let $\{F_n\}$ be a Cauchy sequence in X^*. Then given $\varepsilon > 0$, there exists an $N \geq 1$ such that

$$\|F_n - F_m\| < \varepsilon \quad \text{whenever} \quad n \geq N \quad \text{and} \quad m \geq N.$$

Fix $x \neq 0$ in X. Let $\varepsilon > 0$ be given, and let $\varepsilon_1 = \varepsilon/\|x\|$. Then there exists an $N \geq 1$, such that n and $m \geq N$ implies

$$\|F_n - F_m\| < \frac{\varepsilon}{\|x\|}.$$

Using the linearity of X^*, and the definition of $\|F_n - F_m\|$, we have

$$|F_n(x) - F_m(x)| = |(F_n - F_m)\, x| \leq \|F_n - F_m\| \; \|x\| < \varepsilon$$

for $n,m \geq N$. Hence $\{F_n(x)\}$ is a Cauchy sequence of complex numbers for each fixed $x \neq 0$. The corresponding assertion for $x = 0$ is immediate. Thus, by completeness of the complex number system, there exists a functional F such that

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \text{for each} \quad x \in X.$$

Now

$$F_n(\alpha x + \beta y) = \alpha F_n(x) + \beta F_n(y)$$

for each $x,y \varepsilon X$, all scalars α and β, and for every $n \geq 1$. Hence

$$\lim_{n\to\infty} F_n(\alpha x + \beta y) = \alpha \lim_{n\to\infty} F_n(x) + \beta \lim_{n\to\infty} F_n(y),$$

and F is linear. We must now show that F is bounded, and that

$$\|F_n - F\| \to 0 \quad \text{as} \quad n \to \infty.$$

We know that there exists an N_1 such that

$$\|F_n - F_m\| < 1 \quad \text{for} \quad n,m \geq N_1.$$

Hence

$$\|F_n\| \leq \|F_{N_1}\| + 1 \quad \text{for every} \quad n \geq N_1,$$

and

$$\|F_n\| \leq \max \{ \|F_1\|, \|F_2\|, \cdots, \|F_{N_1 - 1}\|, \|F_{N_1}\| + 1 \} = M$$

for all $n \geq 1$. By the definition of norm of F_n,

$$|F_n(x)| \leq \|F_n\| \|x\| \leq M \|x\|$$

for each $x \varepsilon X$ and each $n \geq 1$. Therefore

$$|F(x)| = \lim_{n\to\infty} |F_n(x)| \leq M\|x\| \text{ for each } x \varepsilon X.$$

Hence

$$\|F\| \leq M < \infty$$

and F is bounded. Now let $\varepsilon > 0$ be given. There exists an N_o such that

$$|F_m(x) - F_n(x)| \leq \|F_n - F_m\| \|x\| < \frac{\varepsilon}{2} \|x\|$$

for $m \geq N_o$ and $n \geq N_o$. Hence

$$|F(x) - F_n(x)| \leq \frac{\varepsilon}{2} \|x\| \text{ for } n \geq N_o$$

and we have

$$\|F - F_n\| \leq \frac{\varepsilon}{2} < \varepsilon \text{ for } n \geq N_o.$$

Thus X^* is complete. This completes the proof.

2-7 *Hahn-Banach Theorem.* If M is a subspace of a normed linear space X, and if F is a bounded linear functional defined on M, then F can be extended to a bounded linear functional G on X in such a way that $\|G\| = \|F\|$.

This theorem is proved in Rudin [8].

2-8 *Corollary.* If X is a Banach space and $x \in X$, $x \neq 0$, then there exists a bounded linear functional $F \in X^*$ such that

$$\|F\| = 1 \quad \text{and} \quad F(x) = \|x\| .$$

<u>Proof</u>. Let $x_o \in X$, $x_o \neq 0$, and consider the subspace M spanned by x_o (i.e., all scalar multiples of x_o). On M, define the functional F_o by

$$F_o(cx_o) = c\,\|x_o\|$$

for each scalar c. Since

$$|F_o(cx_o)| = |c|\,\|x_o\| = \|cx_o\|$$

it follows that

$$\|F_o\| = \sup\{|F_o(cx_o)| : \|cx_o\| \leq 1\} \leq 1.$$

But using the linearity of F_o,

$$\|cx_o\| = |F_o(cx_o)| \leq \|F_o\| \, \|cx_o\| ,$$

and thus $\|F_o\| \geq 1$. Therefore $\|F_o\| = 1$. Applying the Hahn-Banach theorem, there exists a bounded linear functional F defined on the entire space X such that F is an extension of F_o, and $\|F_o\| = \|F\|$. Hence

$$\|F\| = 1 \quad \text{and} \quad F(x_o) = \|x_o\| ,$$

and the corollary is proved.

The result guarantees that if $X \neq \{0\}$ then the dual space X^* does not consist of the zero element.

2-9 $L(X^*,F)$, the set of all bounded linear functionals defined on X^*, is called the *second dual* of X, and is written X^{**}.

Now for each fixed x, define Λ_x on X^* as follows:

$$\Lambda_x(F) = F(x) \quad \text{for each} \quad F \in X^* .$$

The linear functional Λ_x is in X^{**} for each $x \in X$. In fact, it will be shown that

$$\|\Lambda_x\| = \|x\| \quad \text{for each } x,$$

where

$$\|\Lambda_x\| = \sup\{|\Lambda_x(F)| : \|F\| \le 1\} .$$

That $\|\Lambda_x\| \le \|x\|$ follows easily from the definition of norm of a transformation, since

$$|\Lambda_x(F)| = |F(x)| \le \|F\| \, \|x\| \le \|x\|$$

if $\|F\| \le 1$. To show that $\|\Lambda_x\| \ge \|x\|$, suppose first that $x \ne 0$. Then by Corollary 1-8 there exists a bounded linear functional $G \varepsilon X^*$ such that

$$\|G\| = 1 \quad \text{and} \quad G(x) = \|x\| .$$

Hence for this G we have

$$\Lambda_x(G) = G(x) = \|x\| .$$

Since $\|G\| = 1$,

$$\|x\| = |\Lambda_x(G)| \le \|\Lambda_x\| \, \|G\| = \|\Lambda_x\| .$$

Therefore

$$\|x\| \le \|\Lambda_x\| \quad \text{if} \quad x \ne 0 .$$

Since it is clear that $\|\Lambda_o\| = \|0\| = 0$, we know that

$$\|x\| \le \|\Lambda_x\| \quad \text{for all } x \in X \, .$$

Hence we have shown that $\|\Lambda_x\| = \|x\|$ and $\Lambda_x \in X^{**}$ for each $x \in X$.

2-10 The mapping $x \to \Lambda_x$ is called the *natural mapping* of X into X^{**} (since norms are preserved, this mapping is an isometry).

A Banach space is *reflexive* if the natural mapping maps X *onto* X^{**}. Some examples of reflexive Banach spaces are

(i) All finite dimensional Banach spaces

(ii) ℓ^p spaces for $1 < p < \infty$

(iii) $L^p(\mu)$ spaces for $1 < p < \infty$ (μ is a positive measure on a σ-algebra of subsets of a space).

The spaces $L^1(\mu)$, ℓ^1, $L^\infty(\mu)$, ℓ^∞, and $C^r[a,b]$, where $[a,b]$ is a bounded interval in the real line, are Banach spaces which are *not* reflexive. For proofs of these assertions reference may be made to Rudin [8].

2-11 If $\{x_n\}$ is a sequence of elements of the Banach space X, we say that $\{x_n\}$ *converges weakly* to an element $x \in X$ if

$$G(x_n) \to G(x) \quad \text{for every} \quad G \in X^* .$$

Since

$$|G(x_n) - G(x)| = |G(x_n - x)| \le \|G\| \, \|x_n - x\| \, ,$$

with $\|G\| < \infty$ for each $G \in X^*$, we see that convergence in the norm of X implies weak convergence in X. Therefore convergence in norm is "stronger" than weak convergence, and this is one reason that the term "strong" is often used for convergence in norm.

For an example of a sequence of elements in a Banach space which converges weakly to some element in that space, but does not converge in norm (strongly), we consider ℓ^2, the Banach space consisting of all sequences $x = \{\xi_i\}$, $i = 1,2,3,\cdots$, of complex numbers such that

$$\|x\| = (\sum_{i=1}^{\infty} |\xi_i|^2)^{1/2} < \infty .$$

It is known (see Lorch [6], for example) that the dual space of ℓ^2 is ℓ^2, that is, every bounded linear functional F defined on ℓ^2 can be expressed uniquely by means of a sequence $\{f_i\}$ of complex numbers such that

$$\|F\| = (\sum_{i=1}^{\infty} |f_i|^2)^{1/2} < \infty ,$$

The components f_i are such that

$$F(x) = \sum_{i=1}^{\infty} f_i \, \xi_i$$

for each $x = \{\xi_i\}$ in X.

Now let $\{e_n\}$ be a sequence of elements in ℓ^2 defined as follows:

$$e_1 = (1,0,0,\cdots), \quad e_2 = (0,1,0,\cdots), \quad \cdots$$

$$\cdots, e_n = (0,0,\cdots 0,1,0,\cdots), \cdots$$

where e_n has 1 in the nth place and zeroes elsewhere.

Then

$$\|e_n - e_m\| = \sqrt{2} \quad \text{for} \quad n \neq m ,$$

and hence $\{e_n\}$ cannot converge in norm. But now suppose F, represented by $(f_1,f_2,\cdots)$, is a bounded linear functional defined on ℓ^2.

We have

$$F(e_n) = \sum_{i=1}^{\infty} \delta_{in} f_i = f_n \text{ for each } n .$$

$$(\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} , \text{ called the } \textit{Kronecker delta})$$

Since $\sum_{i=1}^{\infty} |f_i|^2 < \infty$, we know that

$$f_n \to 0 \quad \text{as} \quad n \to \infty .$$

Hence

$$F(e_n) \to 0 = F(0) \quad \text{as } n \to \infty ,$$

and we have shown that $\{e_n\}$ converges weakly to the zero element of ℓ^2.

Weak convergence of $\{x_n\}$ to x is often indicated by the notation

$$x_n \xrightarrow{w} x .$$

Likewise, convergence of $\{x_n\}$ to x in norm is indicated by

$$x_n \xrightarrow{s} x .$$

2-12 Theorem. If $x_n \xrightarrow{w} x$ in X, and $x_n \xrightarrow{w} y$ in X, then $x = y$. In other words, weak limits are unique (as are limits in norm).

Proof. Suppose $x \neq y$. Then $x - y \neq 0$, and by Corollary 1-9, there exists a bounded linear functional $F \in X^*$ such that

$$\|F\| = 1 \text{ and } F(x - y) = \|x - y\| ,$$

Since $\|x - y\| > 0$, we have

$$F(x) - F(y) = F(x - y) > 0 .$$

Therefore $F(x) \neq F(y)$. Now by the definition of weak convergence,

$$\lim_{n\to\infty} G(x_n) = G(x), \text{ and } \lim_{n\to\infty} G(x_n) = G(y)$$

for each $G \in X^*$. Since limits of sequences of complex numbers are unique, we must have $G(x) = G(y)$ for *each* $G \varepsilon X^*$. But the assumption that $x \neq y$ leads to a contradiction of this fact. Hence $x = y$. This completes the proof.

2-13 If $\{F_n\}$ is a sequence in X^*, and $F \varepsilon X^*$, to say that

$$F_n \to F \text{ weakly in } X^*$$

means, in accordance with the definition in 1-9, that

$$\Lambda(F_n) \to \Lambda(F) \text{ for all } \Lambda \varepsilon X^{**} \tag{1}$$

Since this requirement is often too strong for use in applications, it is useful to require (1) to be true only for that *subset* of X^{**} which corresponds to X under the natural mapping of X into X^{**}. Thus we replace (1) by

$$\Lambda_x(F_n) \to \Lambda_x(F) \quad \text{for all } x \varepsilon X .$$

But this is equivalent to requiring that

$$F_n(x) \to F(x) \quad \text{for all } \; x \varepsilon X .$$

We have given the motivation for the definition of another type of convergence in X^*: If X is a normed linear space, $\{F_n\}$ is a sequence in X^*, and $F \varepsilon X^*$, then $\{F_n\}$ is said to converge *weakly** to F if

$$\lim_{n\to\infty} F_n(x) = F(x) \text{ for all } x \in X .$$

Note that if X is reflexive, then weak* convergence and ordinary weak convergence are the same.

2-14 Suppose X and Y are normed linear spaces, and that $T \in L(X,Y)$. Let $G \in Y^*$. For each $x \in X$, define F on X by

$$F(x) = G(Tx) .$$

The functional F is linear, by the linearity of G and T. Also

$$\|F(x)\| = \|G(Tx)\| \le \|G\| \, \|Tx\| \le \|G\| \, \|T\| \, \|x\|$$

for each $x \in X$, since G and T are bounded. Therefore $F \in X^*$. To designate the F so associated with G (by the fixed transformation T), we write

$$F = T^* G .$$

Observe that T^* maps Y^* into X^*, whereas T maps X into Y. The transformation T^* is called the *adjoint* (or *conjugate*) *of T.*

2-15 *Theorem.* If T is a bounded linear transformation of X into Y, and T^* is the adjoint of T, then T^* is a bounded linear transformation of Y^* into X^*, and

$$\|T^*\| = \|T\| .$$

Proof. By definition,

$$T^*G(x) = F(x) = G(Tx) \text{ for each } x \in X .$$

To show that T^* is linear, let G and H be in Y^*, and let α and β be scalars. For every $x \in X$,

$$\{T^*(\alpha G + \beta H)\}(x) = \{\alpha G + \beta H\}\ (Tx) =$$

$$\alpha G(Tx) + \beta H(Tx) = \alpha T^*G(x) + \beta T^*H(x) =$$

$$\{\alpha T^*G + \beta T^*H\}\ (x) .$$

Thus T^* is linear on Y^*. Next we show $\|T^*\| \leq \|T\|$, which implies that T^* is bounded. By definition of norm,

$$\|T^*\| = \sup\{\ \|T^*G\| : \|G\| \leq 1\ \} .$$

Now

$$\|T^*G\| = \sup\{|T^*G(x)| : \|x\| \leq 1\} = \sup\{|G(Tx)| : \|x\| \leq 1\} ,$$

and

$$|G(Tx)| \le \|G\| \, \|T\| \, \|x\| \text{ for each } x \in X .$$

Hence

$$|G(Tx)| \le \|G\| \, \|T\| \quad \text{whenever} \quad \|x\| \le 1 .$$

It follows that

$$\|T^{*}G\| \le \|G\| \, \|T\| \quad \text{for all } G \in Y^{*}$$

and hence

$$\|T^{*}\| \le \|T\| .$$

It remains to prove $\|T^{*}\| \ge \|T\|$. Let $x \in X$. If $Tx \neq 0$, by Corollary 1-9 there exists a $G \in Y$ such that

$$\|G\| = 1 \quad \text{and} \quad G(Tx) = \|Tx\| .$$

Then $\|Tx\| = |G(Tx)| = |T^{*}G(x)| \le \|T^{*}G\| \, \|x\| \le \|T^{*}\| \, \|G\| \, \|x\| = \|T^{*}\| \, \|x\|$.

If $Tx = 0$, then

$$\|Tx\| \le \|T^{*}\| \quad \|x\|$$

holds trivially. Hence

$$\|T\| \leq \|T^*\| ,$$

and we conclude that $\|T\| = \|T^*\|$. The proof is complete.

2-16 It is useful to write

$$T^*G(x) = G(Tx)$$

in the "inner product" notation

$$(x,T^*G) = (Tx,G) \text{ for any } x \in X,\ G \in Y^* .$$

All of the operations associated with inner products in Hilbert spaces are valid for this notation, and justification for naming T^* the "adjoint" of T is apparent.

2-17 Suppose we now let $T \in L(X,X)$ and $G \in X^*$. Then the transformation T^* defined by

$$(x,T^*G) = (Tx,G) \text{ for all } x \in X \tag{2}$$

maps X^* into X^*.

<u>Theorem</u>. Let $T \in L(X,X)$ where X is a normed linear space, and let $G \in X^*$. Then for all $x \in X$,

$$(x, (T^*)^nG) = (T^nx,G) \text{ for each } n \geq 1 .$$

<u>Proof</u>. (i) For $n = 1$, we have the definition of T^*.

(ii) Suppose for some $k \geq 1$ we have

$$(x,(T^*)^k G) = (T^k x, G) \ . \tag{3}$$

Now

$$(x,(T^*)^{k+1} G) = (x,(T^*)^k T^* G) \ .$$

Using (3), and the fact that $T^* G \in X^*$, we have

$$(x,(T^*)^k T^* G) = (T^k x,\ T^* G) \ .$$

By (2),

$$(T^k x,\ T^* G) = (T(T^k x),\ G) \ .$$

Hence

$$(x,(T^*)^{k+1} G) = (T^{k+1} x, G) \ ,$$

and the proof by induction is complete.

The notion of weak* convergences defined in 1-13, can be formulated in inner product notation: a sequence $\{F_n\}$ in X^* converges weakly* to $F \in X^*$ if

$$(x, F_n) \to (x, F) \quad \text{as} \quad n \to \infty$$

for every $x \in X$.

CHAPTER III

THEOREMS FOR BANACH SPACES

3-1 Let X be a Banach space, let $T \varepsilon L(X,X)$, and suppose $f \varepsilon X$. We will supply conditions that are sufficient for solution of the equation

$$u - Tu = f \tag{1}$$

by the method of successive approximations with the *Picard (Poincaré, Neumann)* iterative scheme:

$$x_{n+1} = Tx_n + f \quad \text{for} \quad n > 0,$$

where where x_o is some given element in X. It is clear that the nth Picard iterate, x_n, is given by

$$x_n = \sum_{i=1}^{n} Tf^{i-1} + T^n x_o, \tag{2}$$

where $T^o = I$. Then our problem is that of finding conditions sufficient for convergence of the series whose nth partial sums are given by (2).

3-2 A bounded linear operator T on a Banach space X is said to be *asymptotically convergent* if, for each $x \varepsilon X$, the sequence of elements

$\{T^k x\}$ converges in norm to some element $y \in X$. Throughout the rest of the chapter, the term *operator* will refer to a bounded linear operator.

An operator T on a Banach space X is *asymptotically bounded* if there exists a real number $M > 0$ such that

$$\|T^n\| \leq M \quad \text{for all} \quad n \geq 1.$$

Now if T is asymptotically convergent, it follows from the Banach-Steinhaus theorem applied to $\{T^n\}$ that there exists a real number $M > 0$ such that $\|T^n\| \leq M$ whenever $n \geq 1$. Hence T is also asymptotically bounded. In addition, it was shown in 2-5 that there exists a bounded linear operator Q such that

$$\|T^n x - Qx\| \to 0 \quad \text{as} \quad n \to \infty .$$

3-3 *Theorem* (Browder-Petryshyn). Let X be a Banach space, let $T \in L(X,X)$, and suppose T is asymptotically convergent. Then:

(a) If f is an element in the range of the operator $I - T$ (that is, if a solution of (1) exists), the sequence $\{x_n\}$ defined in (2) for any initial approximation x_o will converge to a solution of the equation

$$u - Tu = f.$$

(b) If any subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges to an element y of X, then y is a solution of

$$y - Ty = f$$

and the whole sequence $\{x_n\}$ converges to y.

<u>Proof</u>. Since T is asymptotically convergent, we know from Section 3-2 that there exists an operator Q such that

$$\|T^n x - Qx\| \to 0 \quad \text{as} \quad n \to \infty$$

for each $x \in X$. Now by 2-11, we see that for each fixed x in X, the sequence of elements $\{T^n x\}$ in X converges weakly to Qx; that is,

$$G(T^n x) \to G(Qx) \quad \text{as} \quad n \to \infty$$

for every $G \in X^*$, or in inner product notation,

$$(T^n x, G) \to (Qx, G) \quad \text{as} \quad n \to \infty \tag{3}$$

for every $G \in X^*$. Applying Theorem 2-17, we have

$$(x, (T^*)^n G) = (T^n x, G)$$

for each $G \in X^*$, and for all $n \geq 1$. By definition of the adjoint of an operator in L(X,X),

$$(Qx,G) = (x,Q^*G).$$

Hence (3) can be written in the form

$$(x,(T^*)^nG) \to (x,Q^*G) \quad \text{as} \quad n \to \infty \ . \tag{4}$$

Since (4) holds for each $x \in X$, we have shown that the sequence $\{(T^*)^nG\}$ in X^* converges weakly* to Q^*G for each $G \in X^*$. Now for each $x \in X$ and $n \geq 1$,

$$(T^nx,(T^*)^nG) - (Qx,Q^*G) = (T^*)^nG(T^nx - Qx) + (T^*)^nG(Qx) - Q^*G(Qx) \ .$$

By Theorem 2-17, we know $(T^n)^* = (T^*)^n$. Thus

$$\|(T^*)^n\| = \|(T^n)^*\| = \|T^n\| \leq M < \infty$$

for $n \geq 1$, since T is asymptotically bounded. Consequently

$$|(T^*)^nG(T^nx - Qx)| \leq M \|G\| \|T^nx - Qx\| \to 0$$

as $n \to \infty$. In view of (4), with Qx in place of x,

$$(T^*)^nG(Qx) \to Q^*G(Qx)$$

as $n \to \infty$. Therefore for each $x \in X$ and for each $G \in X^*$,

$$(T^n x, (T^*)^n G) \to (Qx, Q^* G) \quad \text{as} \quad n \to \infty . \tag{5}$$

Since

$$(T^{2n} x, G) = (T^n x, (T^*)^n G)$$

by Theorem 2-17, and since

$$(Qx, Q^* G) = (Q^2 x, G),$$

we have, by substitution in (5),

$$(T^{2n} x, G) \to (Q^2 x, G) \quad \text{as} \quad n \to \infty .$$

In other words, the sequence $\{T^{2n}x\}$ in X converges weakly to Q^2x. Now $\{T^{2n}x\}$ is a subsequence of $\{T^n x\}$. Clearly, if a sequence of elements of X converges weakly, then every subsequence converges weakly also, and to the same limit. Therefore $\{T^{2n}x\}$ must converge weakly to Qx. But by Theorem 2-12, weak limits are unique. Hence we have shown that

$$Qx = Q^2 x \quad \text{for all} \quad x \in X .$$

Now we have

$$\|T^{n+1}x - QTx\| = \|T^n(Tx) - Q(Tx)\| \to 0 \quad \text{as } n \to \infty$$

for each $x \in X$. Since also

$$\|T^{n+1}x - Qx\| \to 0 \quad \text{as } n \to \infty ,$$

we have, by the uniqueness of strong limits, that

$$QTx = Qx \quad \text{for all} \quad x \in X .$$

In addition,

$$\|T^{n+1}x - TQx\| \le \|T\| \, \|T^n x - Qx\| \quad \text{for every} \quad x \in X .$$

Since $\|T\| < \infty$, we know that

$$\|T^{n+1}x - TQx\| \to 0 \quad \text{as} \quad n \to \infty .$$

Hence

$$TQx = Qx \quad \text{for all} \quad x \in X .$$

The results of the proof up to this point are that

$$Q = Q^2 \quad \text{and} \quad Q = QT = TQ . \tag{6}$$

With these results established, we will proceed with the proof of part (a).

Suppose f is in the range of the operator I - T. Then

$$f = u - Tu$$

for some $u \in X$. Define, for $n \geq 1$,

$$S_n(f) = f + Tf + \cdots T^{n-1}f = \sum_{i=1}^{n} T^{i-1}f ,$$

where T^o represents the identity operator I. Then the iterates in (2) have the form

$$x_n = S_n(f) + T^n x_o . \tag{7}$$

Now

$$S_n(f) = S_n(u - Tu) = u - Tu + Tu - T^2u + T^2u + \cdots + T^n u$$

$$= u - T^n u .$$

Consequently

$$\|x_n - (u - Q(u - x_o))\| = \|S_n(f) + T^n x_o - u + Qu - Qx_o\|$$

$$= \|T^n x_o - T^n u + Qu - Qx_o\| \leq \|T^n u - Qu\| + \|T^n x_o - Qx_o\| ,$$

and it follows that

$$\|x_n - (u - Q(u - x_o))\| \to 0 \quad \text{as} \quad n \to \infty .$$

We will now show, by using (6), that $u - Q(u - x_o)$ is a solution of (1). This follows by observing that

$$[u - Q(u - x_o)] - T[u - Q(u - x_o)] =$$

$$u - Qu + Qx_o - Tu + TQu - TQx_o =$$

$$u - Qu + Qx_o - Tu + Qu - Qx_o = u - Tu = f .$$

The proof of part (a) is complete.

To prove (b), suppose $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$, with x_n as defined in (7), which converges to y in the norm of X:

$$x_{n_j} = T^{n_j} x_o + S_{n_j}(f) ,$$

and

$$x_{n_j} \xrightarrow{s} y .$$

It is clear that the subsequence $\{T^{n_j}x\}$ of $\{T^n x\}$ in X will converge to Qx in norm. Observing that

$$\|S_{n_j}(f) - (y - Qx_o)\| = \|x_{n_j} - T^{n_j} x_o - y + Qx_o\|$$

$$\leq \|x_{n_j} - y\| + \|T^{n_j} x_o - Qx_o\| ,$$

it follows that

$$\|S_{n_j}(f) - (y - Qx_o)\| \to 0 \quad \text{as} \quad n \to \infty .$$

Since by (6),

$$QS_{n_j}(f) = Qf + QTf + \cdots + QT^{n_j-1}f$$

$$= Qf + Qf + \cdots + Qf = n_j Qf ,$$

and

$$QT^{n_j} x_o = QTT^{n_j-1} = QT^{n_j-1} = \cdots = Qx_o ,$$

we have

$$Qx_{n_j} = QT^{n_j} x_o + QS_{n_j}(f) = Qx_o + n_j Qf .$$

Therefore

$$\|Qf\| = \left\|\frac{1}{n_j} (Qx_{n_j} - Qx_o)\right\| \leq \frac{\|Q\|}{n_j} \|x_{n_j} - x_o\| ,$$

where we have used some of the basic properties of norms. Now since $x_{n_j} \xrightarrow{s} y$, we see that given any $\varepsilon > 0$, we can be assured that

$$\|Qf\| < \varepsilon .$$

Hence we have

$$Qf = \|Qf\| = 0 .$$

Moreover,

$$(I - T)S_{n_j}(f) = (I - T)(f + \cdots + T^{n_j-1} f)$$

$$= f + Tf + \cdots + T^{n_j-1} f - Tf + \cdots + T^{n_j} f$$

$$= f - T^{n_j} f , \quad \text{for each } j .$$

Now since

$$S_{n_j}(f) \xrightarrow{s} y - Qx_o ,$$

and I - T is bounded, and hence continuous, we see that

$$\|(I - T)S_{n_j}(f) - (I - T)(y - Qx_o)\| \to 0 \quad \text{as} \quad j \to \infty .$$

But by (6),

$$(I - T)(y - Qx_o) = y - Qx_o - Ty + TQx_o = (I - T)y \ .$$

Then

$$(I - T)S_{n_j}(f) - (I - T)(y - Qx_o) = f - T^{n_j} f - (I - T)y \ ,$$

and

$$\|(I - T)y - f\| \le \|(I - T)S_{n_j}(f) - (I - T)(y - Qx_o)\| + \|T^{n_j} f\|$$

for each j. Since

$$\|T^{n_j} f\| = \|T^{n_j} f - Qf\| \to 0 \quad \text{as} \quad j \to \infty \ ,$$

it follows that

$$\|(I - T)y - f\| = 0 \ .$$

This implies that

$$(I - T)y = f \ ,$$

thus showing that y is a solution of (1). We see also that since y is a solution of (1), part (a) of the theorem guarantees that the entire sequence $\{x_n\}$ defined in (7) converges to y.

Before proceeding further, it will be necessary to give a preliminary discussion of a part of the theory of Banach spaces used in the next major result.

3-4 Let X be a non-empty set. A class $\mathcal{J}$ of subsets of X is a *topology* on X if $\mathcal{J}$ satisfies the conditions:

(i) The empty set ϕ and the space X are members of $\mathcal{J}$,

(ii) The union of every class of sets in $\mathcal{J}$ is a set in $\mathcal{J}$,

(iii) The intersection of every finite class of sets in $\mathcal{J}$ is a set in $\mathcal{J}$.

A *topological space,* denoted $(X,\mathcal{J})$, consists of a nonempty set X and a topology $\mathcal{J}$ in X. The sets in $\mathcal{J}$ are called the *open sets* of the topological space. When the topology $\mathcal{J}$ is understood, it is usual to speak of "the topological space X" for brevity. A *closed set* in a topological space X is a set whose complement relative to X is an open set, i.e., a set in the topology $\mathcal{J}$. If A is a subset of the topological space X, the *closure* of A, denoted by $\bar{A}$, is the intersection of all closed sets in X which contain A. Note that it follows from the de Morgan formulas that the intersection of an arbitrary collection of closed sets in X is a closed set in X. Thus $\bar{A}$ is the minimal closed set in X which contains A, and A is a closed set if and only if $A = \bar{A}$.

3-5 A *Hausdorff space* is a topological space with the property that whenever x and y are distinct points of X, there exist disjoint open sets V and W in X such that $x \in V$ and $y \in W$.

If X is a topological space, and if $x \in X$ and $x_n \in X$ for

$n = 1,2,\cdots$, the sequence $\{x_n\}$ is said to *converge to x in the topology on X* if, for each open set V containing x, there exists an index N such that $x_n \in V$ for all $n \geq N$. If X is a Hausdorff space and if $\{x_n\}$ converges to x in the topology $\mathcal{T}$ and converges to y in the topology $\mathcal{T}$, then $x = y$, and we can speak of x as the limit of $\{x_n\}$ in this case. This assertion follows easily from the definitions. See, for example, Lorch [6] or Simmons [9].

If X and Y are topological spaces and if f is a mapping (i.e., a function) of X into Y, f is a *continuous* mapping if the inverse image

$$f^{-1}(V) = \{x : x \in X, f(x) \in V\}$$

is an open set in X whenever V is an open set in Y.

3-6 If X is a Banach space, the *strong topology* $\mathcal{T}_s$ on X is the class of all subsets V of X which have the property that, for each $y \in V$, there exists a real number $r > 0$ (dependent on y) such that the ball

$$B(x,r) = \{x : x \in X, \|x - y\| < r\} \subset V .$$

It is easy to verify that the class of sets V specified in this definition satisfies the requirements listed in Section 3-4 for a topology on X. The strong topology on X is also sometimes called the *metric topology* on X, since the standard metric d on a Banach space is that defined by

$$d(x,y) = \|x - y\| .$$

If X is a Banach space, the *weak topology* on X will now be defined. Given $x_o \varepsilon X$, $\varepsilon > 0$, $n \geq 1$, and elements $F_1, \cdots, F_n$ of the dual space X^* (the set of all bounded linear functionals on X), let

$$V(x_o, F_1, \cdots, F_n, \varepsilon) = \{x : x \varepsilon X, \ |F_i(x) - F(x_o)| < \varepsilon, \ i = 1, \cdots, n\} .$$

Let U be the class of *all* sets of the form $V(x_o, F_1, \cdots, F_n, \varepsilon)$, where $x_o \varepsilon X$, $\varepsilon > 0$, and $F_1, \cdots F_n \varepsilon X^*$ for some positive integer n. Let $\mathcal{J}_w$ be the class of all sets which are unions of subclasses of U. That is, a set V belongs to $\mathcal{J}_w$ if and only if $V = \bigcup_\alpha V_\alpha$, where each $V_\alpha \varepsilon U$, and α runs through any indexing set. It can be proved, but we shall not do so here, that $\mathcal{J}_w$ so specified is a topology on X in the sense of Section 3-4. It can also be proved that X with the weak topology $\mathcal{J}_w$ is a Hausdorff space. The arguments rest upon the fact noted in Section 2-12 that if x_o and y_o are in X, with $x_o \neq y_o$, there exists an $F \varepsilon X^*$ such that $F(x_o) \neq F(y_o)$.

3-7 *Theorem.* Let X be a Banach space, let $\{x_n\}$ be a sequence of elements of X, and let $x \varepsilon X$. Then:

(a) The sequence $\{x_n\}$ converges strongly (in norm) to x if and only if it converges to x in the strong topology (denoted by $\mathcal{J}_s$) on X.

(b) The sequence $\{x_n\}$ converges weakly to x if and only if it converges to x in the weak topology ($\mathcal{J}_w$) on X.

Proof of (a).

(i) Suppose $x_n \xrightarrow{s} x$. Let V be a set in the strong topology $\mathcal{J}_s$ which contains x. Then there exists a real $r > 0$ such that

$$B(x,r) \subset V .$$

Since $x_n \xrightarrow{s} x$, there exists an $N_1 \geq 1$ such that

$$\|x_n - x\| < r \quad \text{whenever} \quad n \geq N_1 .$$

Therefore $x_n \in V$ for all $n \geq N_1$. Hence x_n converges to x in the strong topology on X.

(ii) Suppose $\{x_n\}$ converges to x in the strong topology. Let $\varepsilon > 0$ be given. The set $B(x,\varepsilon)$ is a set in $\mathcal{J}_s$ which contains x. By hypothesis we know that there exists an $N_2 \geq 1$ such that x_n is in $B(x,\varepsilon)$ if $n \geq N_2$. Then

$$\|x_n - x\| < \varepsilon \quad \text{whenever} \quad n \geq N_2 .$$

Hence $x_n \xrightarrow{s} x$.

Proof of (b).

(i) Suppose $x_n \xrightarrow{w} x$. Let V be a set in the weak topology $\mathcal{J}_w$ which contains x. Using the definition of $\mathcal{J}_w$, there exists a set $V(x_o,F_1,\cdots,F_m,\varepsilon)$ such that

$$x \in V(x_o,F_1,\cdots,F_m,\varepsilon) \subset V \ .$$

Now let

$$\delta = \varepsilon - \max_{1\le i\le m} |F_i(x) - F_i(x_o)|$$

and note that $\delta > 0$, since each $|F_i(x) - F_i(x_o)| < \varepsilon$, and only finitely many inequalities are involved. If $y \in V(x,F_1,\cdots,F_m,\delta)$, then

$$|F_i(y) - F_i(x)| < \delta \quad (i = 1,\cdots,m) \ .$$

Hence, for $i = 1,\cdots,m$,

$$|F_i(y) - F_i(x_o)| \le |F_i(y) - F_i(x)| + |F_i(x) - F_i(x_o)|$$

$$< \delta + \max_{1\le i\le m} |F_i(x) - F_i(x_o)| = \varepsilon \ .$$

This holds for every $y \in V(x,F_1,\cdots,F_m,\delta)$, and thus

$$V(x,F_1,\cdots,F_m,\delta) \subset V(x_o,F_1,\cdots,F_m,\varepsilon) \subset V \ ,$$

where δ is the positive number defined above. Since $x_n \xrightarrow{w} x$, there exists an integer $N_3 \ge 1$ such that

$$|F_i(x_n) - F_i(x_o)| < \delta \qquad (i = 1,\cdots,m)$$

if $n \geq N_3$. Thus $n \geq N_3$ implies that $x_n \in V$.

(ii) Suppose $\{x_n\}$ converges to x in the weak topology. Let $\varepsilon > 0$ be given. For each $F \in X^*$, the set

$$V = V(x,F,\varepsilon)$$

is in $\mathcal{T}_w$ and contains x. Thus there exists an index $N_4 \geq 1$ such that $x_n \in V$ for all $n \geq N_4$. Therefore

$$|F(x_n) - F(x)| < \varepsilon \quad \text{for all} \quad n \geq N_4 .$$

Hence $F(x_n) \to F(x)$ as $n \to \infty$. This holds for each $F \in X^*$. Thus $x_n \xrightarrow{w} x$.

This completes the proof.

3-8 Let $(X,\mathcal{T})$ be a topological space. A class $\{V_i\}$ of open sets in X is said to be an *open cover* of X if each point in X belongs to at least one V_i. A subclass of an open cover which is itself an open cover of X is called a *subcover*. A *compact space* is a topological space in which every open cover has a subcover which consists of a finite number of sets.

3-9 *Theorem.* Let $(X,\mathcal{T})$ be a compact topological space, let (Y,S) be an arbitrary topological space, and let f be a continuous mapping of X into Y. If

$$f(X) = \{y : y \in Y, y = f(x) \text{ for some } x \in X\} ,$$

and S' is the class of subsets of Y formed by intersection of sets in the class S with f(X), then (f(X),S') is a compact topological space.

This theorem is proved in Simmons [10]. We might add that (f(X),S') is a topological space without compactness of $(X,\mathcal{T})$. At times when the meaning is clear, we will refer to the topological space $(X,\mathcal{T})$ as "X." The statement "the space X is compact" shall imply the existence of a topology $\mathcal{T}$ such that $(X,\mathcal{T})$ is compact.

3-10 Theorem. If a Banach space X is compact (with respect to any topology), then there exists a real number M > 0 such that

$$\|x\| \leq M \quad \text{for all} \quad x \in X .$$

Proof. Let $G \in X^*$. Since G is a continuous mapping from X into the scalar field K, Theorem 3-9 applies, and the set

$$G(X) = \{a : a \in K, G(x) = a \quad \text{for some} \quad x \in X\}$$

is a compact set in K. A set is compact in the complex field if and only if the set is closed and bounded. Thus there exists a real number $R_G > 0$ such that

$$|a| \leq R_G \quad \text{for every} \quad a \in G(X) ,$$

and hence

$$|G(x)| \leq R_G \quad \text{for every} \quad x \in X .$$

(This holds for each $G \in X^*$.)

Now the space

$$\mathcal{F} = \{\Lambda_x \in X^{**} : \Lambda_x(F) = F(x) \quad \text{for each} \quad F \in X^*\}$$

is a Banach space. We have

$$|\Lambda_x(F)| = |F(x)| \leq R_F \quad \text{for all} \quad x \in X .$$

Hence

$$\sup_{\Lambda_x \in \mathcal{F}} |\Lambda_x(F)| = \sup_{x \in X} |F(x)| \leq R_F .$$

Applying the Banach-Steinhaus theorem (2-4), we see that there exists a number $M > 0$ such that

$$\|\Lambda_x\| \leq M \quad \text{for all} \quad x \in X .$$

But by 2-9, $\|\Lambda_x\| = \|x\|$. Hence we have shown that

$$\|x\| \leq M \quad \text{for all} \quad x \in X ,$$

and the proof is complete.

3-11 *Theorem* (Eberlein-Smulian). Suppose A is a subset of a Banach space X. The following assertions are equivalent:

(a) Every sequence of elements in A has a subsequence which is weakly convergent to an element of X. (In topological language, the above statement reads "the set A is *relatively weakly sequentially compact*." For the sake of brevity, some of the individual terms here will not be defined.)

(b) Every countably infinite subset of A has a limit point in the weak topology on X.

(c) The closure of A in the weak topology on X is compact in the weak topology. (A set that is compact in the weak topology is said to be *weakly compact.)*

This very deep and important theorem is proved in Dunford-Schwartz [3].

3-12 Let X be a Banach space. A *convex set* in X is a nonempty subset S with the property that, if x and y are in S, then

$$z = (1 - t)x + ty \in S$$

for every real number $t \in [0,1]$. The *convex closure* of a subset A of a Banach space X is the intersection of all closed convex sets containing A.

3-13 *Theorem.* (Krein-Smulian). The convex closure of a weakly compact subset of a Banach space is itself weakly compact.

3-14 *Theorem.* A Banach space is reflexive if and only if the set

$$\{x : x \in X, \|x\| \leq R\}$$

is weakly compact for some $R > 0$.

Theorems 3-13 and 3-14 are also proved in Dunford-Schwartz [3]. We are now ready to state and prove a theorem related to Theorem 3-3.

3-15 *Theorem* (Browder-Petryshyn). Let X be a Banach space, and let T be a bounded linear operator which is asymptotically convergent. If X is reflexive and the sequence $\{x_n\}$, where

$$x_{n+1} = Tx_n + f \quad \text{for} \quad n \geq 0 , \tag{8}$$

(and x_o is given), is a bounded sequence, then $\{x_n\}$ converges to a solution of the equation

$$u - Tu = f . \tag{9}$$

As part of the proof of 3-15, we consider two lemmas.

Lemma A. Suppose X is a Banach space, T is a bounded linear operator on X, and x is a point of X. If there exists a weakly compact subset K of X such that the set

$$\{ \sum_{j=0}^{k} T^j x : k = 0,1,\cdots \} \subset K ,$$

then x is in the range of the operator I - T.

Proof of Lemma A: By Theorem 3-10, there exists a real number $M > 0$ that

$$\left\| \sum_{j=0}^{k} T^j x \right\| \leq M, \quad (k = 0,1,\cdots) . \tag{10}$$

Define, for $n = 1,2,\cdots$,

$$x_n = x - \frac{1}{n} \sum_{j=0}^{n-1} T^j x .$$

Note that on account of (10) we have $x_n \xrightarrow{s} x$. It will now be proved that each x_n is in the range of I - T, and that x is in the range of I - T. Define

$$y_i = x + Tx + \cdots + T^{i-1}x = \sum_{\nu=0}^{i-1} T^\nu x, \quad (i = 1,2,\cdots) . \tag{11}$$

Then

$$Ty_i = Tx + \cdots + T^i x ,$$

and hence

$$y_i - Ty_i = x - T^i x . \tag{12}$$

Now

$$x_n = x - \frac{1}{n}\sum_{j=0}^{n-1} T^j x = x - \frac{1}{n}\sum_{j=1}^{n-1} T^j x - \frac{1}{n} x ,$$

and hence, using (12), for $n \geq 2$,

$$x_n = \frac{n-1}{n} x - \frac{1}{n}\sum_{j=1}^{n-1}\left(x - (y_i - T_{y_i})\right) = \frac{1}{n}\sum_{j=1}^{n-1}(y_i - T_{y_i}) = (I-T)\left\{\frac{1}{n}\sum_{j=1}^{n-1} y_i\right\}.$$

It follows that x_n is in the range of $I - T$ for each $n = 1,2,\cdots$, since $x_1 = 0$. Now set $z_1 = 0$, and

$$z_n = \frac{1}{n}\sum_{i=1}^{n-1} y_i, \quad (n = 2,3,\cdots) .$$

It has been shown that $x_n = (I - T)z_n$. The points $\{y_i\}$ are contained in the weakly compact set K, by hypothesis. If y_o is defined to be zero, we see that the point

$$z_n = \sum_{i=0}^{n-1} \frac{1}{n} y_i$$

lies in any convex subset of X which contains the points $\{y_o,\cdots,y_{n-1}\}$. If K is the convex closure of the set

$$K\cup\{0\} = K\cup\{y_o\} ,$$

then $z_n \in K$ for every $n = 1,2,\cdots$, (by the definition in 3-12). Since $K\cup\{0\}$ is the union of two weakly compact sets in X, it is itself weakly compact. Theorem 3-13 then asserts that K is weakly compact. By Theorem 3-11, it follows that K is also relatively weakly sequentially compact. Hence there exists a point $z \in X$ such that $z_{n_i} \xrightarrow{w} z$ for some subsequence $\{z_{n_i}\}$ of $\{z_n\}$. That is,

$$|G(z_{n_i}) - G(z)| \to 0 \quad \text{as} \quad i \to \infty \quad \text{for each} \quad G \in X^*.$$

Now for each $x \in X$, let

$$F(x) = G(Tx), \quad G \in X^*.$$

We have shown in 2-14 that $F \in X^*$ for each $G \in X^*$. Thus for each $G \in X^*$, we have

$$|G(Tz_{n_i}) - G(Tz)| = |F(z_{n_i}) - F(z)| \to \text{as } i \to \infty,$$

and therefore $Tz_{n_j} \xrightarrow{w} Tz$. Then clearly

$$(I - T)z_{n_j} \xrightarrow{w} (I - T)z .$$

But

$$(I - T)z_{n_i} = x_{n_i} ,$$

and consequently $x_{n_j} \xrightarrow{w} (I - T)z$. On the other hand, since $x_n \xrightarrow{s} x$, we have $x_n \xrightarrow{w} x$, (by 2-11) and therefore $x_{n_i} \xrightarrow{w} x$. We have shown in

2-12 that weak limits are unique. Consequently,

$$x = (I - T)z,$$

and hence x is in the range of I - T, as asserted

Lemma B. Suppose X is a reflexive Banach space, T a bounded linear operator on X, and x is a point of X. Then:

(a) If the set

$$S = \{ \sum_{i=0}^{k} T^i x : k = 0,1,2,\cdots \}$$

is strongly bounded; i.e., bounded in the norm of X, then x is in the range of I - T.

(b) If T is asymptotically bounded, and x is in the range of I - T, then the set S above is necessarily strongly bounded.

Proof of Lemma B, Part (a): The set S is contained in the set

$$A = \{x : x \in X : \|x\| \leq R\}$$

for some $R > 0$, since S is strongly bounded. Now by Theorem 3-14, the set A is weakly compact. Thus, by Lemma A, x is in the range of the operator I - T.

<u>Proof</u> of Lemma B, Part (b). By hypothesis, there exists a real number $M > 0$ such that

$$\|T^n\| \leq M \quad \text{for all} \quad n \geq 1 \ .$$

In addition, there exists an element $z \in X$ such that $x = (I - T)z$. Then, using linearity of T,

$$\sum_{j=0}^{k} T^j x = \sum_{j=0}^{k} T^j (I - T) z = z - T^{k+1} z \ .$$

Hence

$$\left\| \sum_{j=0}^{k} T^j x \right\| = \|z - T^{k+1} z\| \leq (1 + M) \|z\| < \infty$$

for all $k \geq 0$, and the assertion is verified. Using the results of Lemmas A and B, we will complete the proof of the theorem.

By hypothesis, X is a reflexive Banach space, and T is asymptotically convergent, i.e., the sequence $\{T^n x\}$ converges strongly in X as $n \to \infty$, for each $x \in X$. As pointed out in Section 3-2, it follows from the Banach-Steinhaus theorem that T is asymptotically bounded.

The nth term in the sequence $\{x_n\}$ defined in (8) can also be written

$$x_n = T^n x_o + \sum_{j=0}^{n-1} T^j f, \quad \text{where} \quad T^o = I \ .$$

Now

$$\left\| \sum_{j=0}^{n-1} T^j f \right\| = \|x_n - T^n x_o\| \le \|x_n\| + \|T^n\| \, \|x_o\| .$$

If $\{x_n\}$ is taken to be bounded in the norm of X, then, since T is asymptotically bounded, the set

$$\{ \sum_{j=0}^{n-1} T^j f : n = 1,2,\cdots\}$$

is strongly bounded. Hence by Lemma B, f is in the range of the operator I - T. Then Theorem 3-3 applies, and the sequence $\{x_n\}$ converges to a solution of (9). The proof is complete.

The next theorem will verify that a certain class of operators will satisfy the hypotheses of Theorems 3-3 and 3-15.

3-16 Theorem. Let T be a bounded linear operator in L(X,X) for which (-1) is not an eigenvalue, and suppose there is an operator Q such that

$$\|T^{2n}x - Qx\| \to 0 \quad \text{as} \quad n \to \infty$$

for each x in X. Then T is asymptotically convergent.

Proof. Since T is bounded and

$$\|(T^2)^n x - Qx\| \to 0 \quad \text{as} \quad n \to \infty$$

for all $x \varepsilon X$, the proof of Theorem 3-3 applies, with T replaced by T^2, from which it follows that $Q \varepsilon L(X,X)$, and

$$QT^2 = T^2Q = Q . \tag{14}$$

The range of an operator Q denoted by R(Q), is the set

$$\{x : x \varepsilon X, \quad x = Ry \quad \text{for some} \quad y \varepsilon X\} .$$

The null space of an operator Q, written N(Q), is the set

$$\{x : x \varepsilon X, Qx = 0\} .$$

We will now prove two useful equalities using set inclusion arguments.

$$(A) : R(Q) = N(I - T^2) .$$

Proof. Let $x \varepsilon N(I - T^2)$. Then

$$x = T^2x = T^4x = \cdots = T^{2n}x \quad \text{for each} \quad n \geq 1 .$$

Therefore

$$x = Qx, \quad \text{and} \quad x \varepsilon R(Q) .$$

(ii) Now let $y \varepsilon R(Q)$. Then $y = Qx$ for some $x \varepsilon X$. Using (14),

$$T^2y + T^2Qx = Qx = y \ .$$

Therefore $y \in N(I - T^2)$. By (i) and (ii), we have

$$R(Q) = N(I - T^2) \ .$$

$$(B) : N(I - T^2) = N(I - T) \ .$$

Proof. (i) Let $x \in N(I - T)$. Then

$$x = Tx = T^2x \ .$$

Hence

$$x \in N(I - T^2) \ .$$

(ii) Now let $x \in N(I - T^2)$. Then $x = T^2x$. We can write

$$Tx = z = x + y \quad \text{for some} \quad y \in X \ .$$

Therefore

$$x = T^2x = Tz + Ty = x + y + Ty \ ,$$

and hence

$$-y = Ty .$$

Since (-1) is not an eigenvalue of T, y = 0, and

$$Tx = x .$$

Hence $x \varepsilon N(I - T)$. By (i) and (ii), (B) is proved. By combining the relations A and B, we have

$$R(Q) = N(I - T) .$$

Hence, if $x \varepsilon X$ and we let $y = Qx$, then

$$Ty = y, \quad \text{or} \quad TQx = Qx .$$

This holds for each $x \varepsilon X$. Therefore

$$\|T^{2n+1}x - Qx\| = \|T(T^{2n}x) - TQx\|$$

$$\leq \|T\| \; \|T^{2n}x - Qx\| \to 0 \quad \text{as} \quad n \to \infty .$$

We now have the sequences $\{T^k x\}_{k_{odd}}$ and $\{T^k x\}_{k_{even}}$ tending in norm to the same limit Qx, for each $x \varepsilon X$. Therefore T converges asymptotically, and the theorem is proved.

3-17 Theorems 3-3 and 3-15 are related to an earlier result of

Browder (see Browder [27]) which guarantees the existence of a solution of (1) for a given element $f \in X$ under the assumptions that the space X is *reflexive,* and that the operator T is *asymptotically bounded,* if and only if the sequence $\{x_n\}$ is bounded for any fixed $x_o \in X$. In Theorem 3-3, no assumption of reflexivity of X is made, but a slightly more restrictive requirement is imposed on T, namely that T be asymptotically convergent. However, under these new hypotheses, we not only know that the iterative scheme (8) converges to a solution of (9) (part (a)), but that the limit of any subsequence of the sequence $\{x_n\}$ defined by (8) will be a solution of (9) (part (b)). The latter result does not follow from the earlier work by Browder, Browder [2].

CHAPTER IV

THEOREMS FOR HILBERT SPACES

4-1 A complex linear space H is an *inner product space, or pre-Hilbert space,* if to each pair of elements $x,y \in H$ there is associated a complex number (x,y) such that, whenever $x,y,z \in H$ and λ is a complex scalar,

(a) $(x,x) \geq 0$ and $(x,x) = 0$ if and only if $x = 0$,

(b) $(x + y,z) = (x,z) + (y,z)$,

(c) $(\lambda x,y) = \lambda(x,y)$,

(d) $(x,y) = \overline{(y,x)}$, where $\overline{(y,x)}$ is the complex conjugate of (y,x).

The scalar (x,y) is called the *inner product* of the elements x,y. It follows easily that

$$(x,\lambda y) = \bar{\lambda}(x,y) \quad \text{and} \quad (x,y + z) = (x,y) + (x,z) .$$

4-2 If for each x in an inner product space H we define

$$\|x\| = (x,x)^{1/2} \tag{1}$$

then, whenever $x,y \in H$,

$$|(x,y)| \leq \|x\| \; \|y\| \; .$$

This basic inequality is the Schwartz inequality, and is proved in any text which deals with Hilbert and pre-Hilbert spaces, as for example Halmos [5], Lorch [6], or Rudin [8]. Using the properties of the inner product in 4-1 and the Schwartz inequality, it is easily proved that the inner product space H is a normed linear space with $\|x\|$ as the norm of x. If the inner product space H, equipped with the norm induced by the inner product as in (1), is *complete* as a normed linear space, i.e., if every Cauchy sequence $\{x_n\}$ in H converges strongly to an element $x \in H$, then H is called a *Hilbert space.* A Hilbert space is thus a Banach space of a special type.

It is useful to point out here the fact that the mappings $x \to (x,y)$, $x \to (y,x)$, and $x \to \|x\|$ (for a fixed $y \in H$) are *continuous* on H, where H is a Hilbert space. These assertions follow easily from the Schwartz inequality and properties of the norm. For a detailed proof, see Rudin [8]. The same reference discusses a number of important examples of Hilbert spaces.

4-3 *Theorem* (F. Riesz). If T is a bounded linear functional on a Hilbert space H, then there is a unique element $y \in H$ such that

$$T(x) = (x,y) \quad \text{for all} \quad x \in H \; .$$

This Riesz representation theorem, of fundamental importance in Hilbert space theory, is proved, for example, in Halmos [5], Rudin [8], and Yosida [10].

4-4 Theorem. Let T be a bounded linear operator on a Hilbert space H. Then there exists a unique bounded linear operator T^* on H with the property that

$$(Tx,y) = (x,T^*y) \quad \text{for all } x \text{ and } y \in H . \tag{2}$$

In addition,

$$\|T\| = \|T^*\| .$$

The operator T^* is called the *Hilbert adjoint of T,* or simply the *adjoint of T* when the meaning is clear.

Proof. If y is a fixed element of H, and

$$F(x) = (Tx,y) \quad \text{for all} \quad x \in H ,$$

then F is a linear functional on H. In addition, since

$$|F(x)| \leq \|Tx\| \, \|y\| \leq \|T\| \|y\| \, \|x\| \tag{3}$$

it follows that F is a bounded linear functional on H. The Riesz representation Theorem (4-3) implies that there exists a unique element y^* of H such that

$$F(x) = (x,y^*) \quad \text{for all} \quad x \in H .$$

Each y in H thus determines a bounded linear functional F (depending on y), and hence a unique y^* in H. Let the mapping T^* be defined on H by

$$T^*y = y^* .$$

Then, by definition,

$$(Tx,y) = (x,T^*y) \quad \text{for all } x \text{ and } y \text{ in } H .$$

We now show that T^* is linear. Consider elements y and z in H, and let

$$G(x) = (Tx,(y + z)^*) ,$$

and, on the other hand,

$$G(x) = (Tx,y) + (Tx,z) = (x,y^*) + (x,z^*) = (x, y^* + z^*) .$$

Thus

$$(y + z)^* = y^* + z^* \quad \text{for all } y \text{ and } z \text{ in } H .$$

By definition of T^*,

$$T^*(y + z) = T^*y + T^*z$$

and T^* is thus linear on H. Now

$$F(x) = (x,y^*) \text{ implies } \|F\| = \|y^*\| .$$

To deduce this fact, note first that

$$|F(x)| \le \|y^*\| \, \|x\|$$

and therefore

$$\|F\| \le \|y^*\| .$$

If $y^* = 0$, then $\|F\| = 0 = \|y^*\|$. If $y^* \neq 0$, it follows from the formula

$$\|F\| = \sup\left\{\frac{|F(x)|}{\|x\|} : x \neq 0\right\}$$

and the observation that

$$F(y^*) = (y^*,y^*) = \|y^*\|^2 ,$$

that $\|y^*\| \le \|F\|$. Thus $\|F\| = \|y^*\|$. By the definition of T^*, and by

(3), it follows that

$$\|T^* y\| = \|y^*\| = \|F\| \leq \|T\| \; \|y\| .$$

Thus T^* is a bounded linear operator with

$$\|T^*\| \leq \|T\| . \tag{4}$$

Suppose, to settle the uniqueness assertion, that $\hat{T}$ is a bounded linear operator on H such that

$$(Tx,y) = (x,\hat{T}y) \quad \text{for all x and y in H} .$$

Then

$$(x,T^* y) - (x,\hat{T}y) = 0 ,$$

and consequently,

$$(x,(T^* - \hat{T})y) = 0 \quad \text{for all x and y in H} .$$

In particular, then

$$((T^* - \hat{T})y, (T^* - \hat{T})y) = \|(T^* - \hat{T})y\|^2 = 0$$

for every y in H. Therefore, $\hat{T} = T^*$.

Since (2) has been established, and therefore

$$(T^*x,y) = (x,Ty) \quad \text{for all } x \text{ and } y \text{ in } H,$$

it follows that T is the (unique) adjoint of T^*. That is,

$$T^{**} = (T^*)^* = T .$$

Hence, by (4),

$$\|T\| = \|(T^*)^*\| \leq \|T^*\| \tag{5}$$

Combining (4) and (5), we have $\|T\| = \|T^*\|$.

4-5 Two elements x and y in a Hilbert space H are called *orthogonal* if $(x,y) = 0$. It should be noted that x is orthogonal to x if and only if $x = 0$. If S is a subset of a Hilbert space H, we denote by $S^\perp$ the totality of elements of H orthogonal to every element of the subset S. Even though S is only a subset of H, it follows easily from the properties of the inner product that $S^\perp$ is a *subspace* of H. The term subspace is used here in the usual linear space sense. In addition, $S^\perp$ is a *closed* subspace, i.e., a closed set in the metric topology of H. If y is any point in the closure of $S^\perp$, there exists a sequence $\{y_n\}$ in $S^\perp$ such that $\|y_n - y\| \to 0$ as $n \to \infty$. For each $x \in S$, and every $n \geq 1$, $(x,y_n) = 0$. By continuity of the inner product in its second argument it follows that

$$(x,y) = \lim_{n\to\infty}(x,y_n) .$$

Consequently, $y \in S^{\perp}$. Thus $S^{\perp}$ is a closed subspace of H. It should also be observed that if S is any subset of H, then

$$S \subset (M^{\perp})^{\perp} = M^{\perp\perp},$$

and that if S and S_1 are subsets of H with $S \subset S_1$, then $S_1^{\perp} \subset S^{\perp}$. These remarks follow at once from the definitions. If M is a closed subspace of H, then the closed subspace $M^{\perp}$ is called the *orthogonal complement* of M, and $M \cap M^{\perp} = \{0\}$.

4-6 *Theorem* (The Projection Theorem). Let M be a closed subspace of a Hilbert space H. For each $x \in H$, there exist unique elements $x' \in M$ and $x'' \in M^{\perp}$ such that

$$x = x' + x'' .$$

The element x' has the property that

$$\|x - x'\| = \inf\{\|x - y\| : y \in M\} ,$$

and is called the *projection* of x on M. If mappings P and Q are defined on H by $Px = x'$ and $Qx = x''$, then P and Q are linear operators such that

$$x = Px + Qx \quad \text{for each } x \in H .$$

The *projection operators* P and Q are bounded linear operators such that $P = P^2$, $Q = Q^2$,

$$(Px,y) = (x,Py), \quad \text{and} \quad (Qx,y) = (x,Qy)$$

for all x and y in H.

For a proof of the projection theorem, see, for example, Halmos [5], Rudin [8], or Yosida [10].

4-7 *Theorem.* (a) If M is a closed subspace of a Hilbert space H, then

$$M = M^{\perp\perp},$$

where $M^{\perp\perp} = (M^{\perp})^{\perp}$. (b) If M is any subspace of a Hilbert space H, and $\bar{M}$ is the closure of M in the metric topology of H, then

$$\bar{M} = M^{\perp\perp}.$$

Proof. (a) It has already been noted that $M \subset M^{\perp\perp}$. Now suppose $x \in M^{\perp\perp}$. By the projection theorem $x = x' + x''$, where $x' \in M$ and $x'' \in M^{\perp}$. Since $x' \in M \subset M^{\perp\perp}$, the fact that $M^{\perp\perp}$ is a subspace of H implies that $x'' = x - x'$ is an element of $M^{\perp\perp}$. Thus $x'' \in M^{\perp\perp} \subset M^{\perp}$ and consequently $x'' = 0$. Hence $x = x' \in M$. Thus $M^{\perp\perp} \subset M$. It follows by double inclusion that $M = M^{\perp\perp}$, and assertion (a) is proved.

(b) If M is a subspace of H, $\bar{M}$ is a closed subspace of H. If $\varepsilon > 0$ is given, and $x, y \in \bar{M}$, there exist elements x_o and y_o in M such that

$$\|x - x_o\| < \varepsilon \text{ and } \|y - y_o\| < \varepsilon .$$

If λ_1 and λ_2 are scalars, then

$$\|\lambda_1 x + \lambda_2 y - (\lambda_1 x_o + \lambda_2 y_o)\|$$

$$\leq |\lambda_1| \ \|x - x_o\| + |\lambda_2| \ \|y - y_o\| \leq (|\lambda_1| + |\lambda_2|)\varepsilon$$

It follows from this observation that $\lambda_1 x + \lambda_2 y \in \bar{M}$. Thus $\bar{M}$ is a closed subspace. Since $M \subset \bar{M}$, $(\bar{M})^{\perp} \subset M^{\perp}$ by a remark in 4-5. Hence

$$M^{\perp\perp} = (M^{\perp})^{\perp} \subset (\bar{M})^{\perp\perp} .$$

Since $\bar{M}$ is a closed subspace, part (a) of this theorem shows that $(\bar{M})^{\perp\perp} = \bar{M}$. Consequently

$$M^{\perp\perp} \subset \bar{M} .$$

The other inclusion relation $\bar{M} \subset M^{\perp\perp}$ follows at once by the observation that $M \subset M^{\perp\perp}$ implies

$$\bar{M} \subset \overline{M^{\perp\perp}} = M^{\perp\perp} ,$$

since $M^{\perp\perp}$ is a closed subspace, as pointed out in 4-5. Hence $\bar{M} = M^{\perp\perp}$.

4-8 *Theorem.* Let T and S be bounded linear transformations on a Hilbert space H, and let T^* and S^* be the adjoints of T and S, respectively. Then:

(a) $T^{**} = T$.

(b) $||T^*T|| = ||T||^2$.

(c) If N(T) is the nullspace of T, and $\overline{R(T^*)}$ is the closure (in H) of the range of T^*, then

$$(N(T))^{\perp} = \overline{R(T^*)} \ .$$

(d) $(TS)^* = S^*T^*$.

Proof of (a). Part (a) was established in the proof of Theorem 4-4.

Proof of (b). Note that if $x \in H$,

$$(T^*T)x = T^*(Tx) \ ,$$

and then

$$\|T^*Tx\| \leq \|T^*\| \|Tx\| \leq \|T^*\| \|T\| \|x\| = \|T\|^2 \|x\|$$

since $\|T\| = \|T^*\|$. It follows that

$$\|T^*T\| \le \|T\|^2 < \infty . \tag{6}$$

Now

$$\|Tx\|^2 = (Tx,Tx) \ge 0 ,$$

and thus by the Schwarz inequality (4-2), and the definition of T^*,

$$\|Tx\|^2 = (Tx,Tx) = (x,T^*Tx) \le \|x\| \, \|T^*Tx\| \le \|x\|^2 \, \|T^*T\| .$$

Consequently, for every x in H,

$$\|Tx\| \le \|T^*T\|^{\frac{1}{2}} \, \|x\| .$$

Hence

$$\|T\| \le \|T^*T\|^{\frac{1}{2}} ,$$

and combining this result with (6), we have $\|T^*T\| = \|T\|^2$.

<u>Proof</u> of (c). Note that

$$N(T) = \{x : x \varepsilon H, Tx = 0\} ,$$

$$R(T^*) = \{x : x \varepsilon H, Ty = x \text{ for some } y \varepsilon H\} .$$

(i) Suppose $x \varepsilon N(T)$. Then $Tx = 0$. Hence

$$(x,T^*y) = (Tx,y) = 0 \text{ for every } y \varepsilon H ,$$

which implies that $x \varepsilon (R(T^*))^\perp$. Therefore $N(T) \subset (R(T^*))^\perp$.

(ii) Suppose $y \varepsilon (R(T^*))^\perp$. Then

$$(x,T^*y) = 0 \text{ for all } y \varepsilon H .$$

Then

$$(Tx,y) = 0 \text{ for all } y \varepsilon H ,$$

which implies $Tx = 0$. Hence $x \varepsilon N(T)$, and $(R(T^*))^\perp \subset N(T)$.

By (i) and (ii) we have

$$N(T) = (R(T^*))^\perp ,$$

and therefore by 4-7,

$$(N(T))^\perp = \{(R(T^*))^\perp\}^\perp = \overline{R(T^*)} .$$

<u>Proof</u> of (d). For any x and y in H we have

$$(x,(TS)^{*}y) = (TSx,y) = (Sx,T^{*}y) = (x,S^{*}T^{*}y) .$$

Hence the proof of Theorem 4-8 is complete.

4-9 In this section we will define some of the terms that will be used in the next sections. Throughout this chapter, we consider only bounded linear operators on a Hilbert space H, and the term *operator* refers to a *bounded linear operator* in all cases. An operator on a Hilbert space H is *self-adjoint* if $T = T^{*}$. Thus if T is a self-adjoint operator on H, we have

$$(x,Ty) = (Tx,y) \quad \text{for every} \quad x,y \in H .$$

It follows easily (by properties of the adjoint) that if T_1 and T_2 are self-adjoint operators on H, so are $T_1 + T_2$ and $\lambda_1 T_1 + \lambda_1 T_2$ where λ_1, λ_2 are any *real* scalars.

By a property of inner products,

$$\overline{(Tx,x)} = (x,Tx) \quad \text{for every} \quad x \in H .$$

Consequently, if T is a self-adjoint operator on H, then

$$\overline{(Tx,x)} = (Tx,x) \quad \text{for every} \quad x \in H ,$$

and, as a result, (Tx,x) is *real* for every $x \in H$. If T is a self-adjoint operator on H such that

$$(Tx,x) \geq 0 \quad \text{for every} \quad x \in H$$

then T is called a *positive* self-adjoint operator.

An operator on a normed linear space is called *non-expanding* if $\|T\| \leq 1$.

4-10 Theorem. Let A be a positive self-adjoint operator on a Hilbert space H. Then for any x and y in H we have

$$|(Ax,y)|^2 \leq (Ax,x)(Ay,y) \ . \tag{7}$$

Relation (7) is usually called the *generalized Schwartz inequality.*

Proof. Let x and y be in H, let λ be a real number, and define

$$h_\lambda = x + \lambda(Ax,y)y \ . \tag{8}$$

Since A is self-adjoint and positive, we have

$$0 \leq (Ah_\lambda h_\lambda) = (Ax + \lambda(Ax,y)Ay, \ x + \lambda(Ax,y)y) \tag{9}$$

$$= (Ax,x) + (\lambda(Ax,y)Ay,x) +$$

$$(Ax, \ \lambda(Ax,y)y) +$$

$$(\lambda(Ax,y)Ay,\ \lambda(Ax,y)y)$$

$$= (Ax,x) + \lambda(Ax,y)(Ay,x) + \lambda\overline{(Ax,y)}(Ax,y) +$$

$$\lambda(Ax,y)\ \lambda\overline{(Ax,y)}(Ay,y)$$

$$= (Ax,x) + 2\lambda|(Ax,y)|^2 +$$

$$\lambda^2|(Ax,y)|^2(Ay,y)\ ,$$

or

$$0 \leq A + 2B\lambda + C\lambda^2\ , \tag{10}$$

where it is clear to which non-negative terms in (9) the letters A, B, and C in (10) refer. If $C \neq 0$, let $\lambda = -B/C$. Then (10) becomes

$$B^2 \leq AC\ .$$

If $C = 0$, then $B = 0$ also, since (10) would be false for

$$\lambda < -A/2B \quad \text{if} \quad B \neq 0\ .$$

Therefore $B^2 \leq AC$ always. That is

$$|(Ax,y)|^4 \le (Ax,x)|(Ax,y)|^2(Ay,y) ,$$

or

$$|(Ax,y)|^2 \le (Ax,x)(Ay,y) .$$

4-11 We can establish a partial ordering in the class of self-adjoint operators on a Hilbert space H as follows: If A and B are two self-adjoint operators on H, let $A \ge B$ mean that

$$(Ax,x) \ge (Bx,x) \quad \text{for all} \quad x \in H . \tag{11}$$

Since we cannot have $A \ge B$ and $A \le B$ simultaneously unless $A = B$, our ordering is well defined. But not all self-adjoint operators are comparable by the relation (11). Hence (11) defines only a *partial* ordering of the class of self-adjoint operators on H (for the details in this argument, see Riesz-Nagy [3]). If A and B are self-adjoint operators on H, condition (11) is equivalent to the condition that the self-adjoint operator $A - B$ be *positive*.

4-12 A self-adjoint operator A on a Hilbert space H has the property that there exist real numbers m and M such that

$$m(x,x) \le (Ax,x) \le M(x,x) \quad \text{for every} \quad x \in H . \tag{12}$$

This assertion follows from the fact that (Ax,x) is real for all $x \in H$, and (since A is a bounded linear operator) the Schwartz inequality implies that

$$|(Ax,x)| \leq \|Ax\|\,\|x\| \leq \|A\|\,\|x\|^2 = \|A\|(x,x) ,$$

and hence that

$$-\|A\|\,(x,x) \leq (Ax,x) \leq \|A\|\,(x,x), \quad \text{for all} \quad x \in H .$$

The smallest number M for which (12) holds is called the *least upper bound* of A; the largest number m for which (12) holds is called the *greatest lower bound* of A. The terminology here is that of Riesz-Nagy [7]. In terms of the order relation defined in 4-11, (12) may be expressed in the form

$$mI \leq A \leq MI ,$$

where I is the identity operator on H.

4-13 Theorem (Riesz-Nagy). Every bounded monotonic sequence of self-adjoint operators $\{B_n\}$ on a Hilbert space H converges strongly to a self-adjoint operator B.

Proof. Let $\{B_n\}$ be a bounded monotonically nondecreasing sequence of self-adjoint operators on H. Then there exist two real numbers m and M, with $m < M$, such that

$$mI \le B_1 \le \cdots \le B_n \le B_{n+1} \le \cdots \le MI . \tag{13}$$

If we let

$$A_n = \frac{1}{M - m}(B_n - mI) \quad \text{for each} \quad n \ge 1 ,$$

we can write (13) as

$$0 \le A_1 \le \cdots \le A_n \le A_{n+1} \le \cdots \le I . \tag{14}$$

For $m < n$, define

$$A_{mn} = A_n - A_m .$$

Then

$$I \ge A_{mn} \ge 0 .$$

Using Theorems 4-2 and 4-10, we have for each fixed $x \in H$,

$$\|A_n x - A_m x\|^4 = \|A_{mn} x\|^4 = (A_{mn} x, A_{mn} x)^2$$

$$\le (A_{mn} x, x)(A_{mn}^2 x, A_{mn} x) = (A_n x - A_m x, x)(A_{mn}^2 x, A_{mn} x)$$

$$\le [(A_n x, x) - (A_m x, x)] \|A_{mn}\|^3 \|x\|^2$$

$$\leq [(A_n x, x) - (A_m x, x)] \ \|x\|^2 ,$$

since $\|A_{mn}\| \leq 1$. By (14), the sequence $\{(A_n x, x)\}$ of real numbers is bounded and nondecreasing, and hence convergent. Therefore the sequence of elements $\{A_n x\}$ in H is a Cauchy sequence in the norm of H. Since H is complete, 2-5 establishes the existence of a bounded linear operator A such that

$$\|A_n x - Ax\| \to 0 \quad \text{as} \quad n \to \infty .$$

This holds for each $x \in H$. For any elements x and y in H, we have

$$(Ax,y) - (x,Ay) = (Ax,y) - (A_n x,y) + (A_n x,y) - (x,Ay)$$

$$= (Ax - A_n x,y) + (x,A_n y - Ay) \leq \|Ax - A_n x\| \, \|y\|$$

$$+ \ \|A_n y - Ay\| \, \|x\| .$$

Since

$$\|A_n x - Ax\| \to 0 \quad \text{and} \quad \|A_n y - Ay\| \to 0 \quad \text{as} \quad n \to \infty ,$$

we have

$$(Ax,y) = (x,Ay) \quad \text{for all } x \text{ and } y \text{ in } H .$$

It follows from the definition of A^* that

$$(x,(A - A^*)y) = 0 \quad \text{for all } x \text{ and } y \text{ in } H ,$$

and we must have

$$((A - A^*)y, \ (A - A^*)y) = 0 \quad \text{for all } y \in H .$$

Hence $A = A^*$, and A is self-adjoint. The original sequence $\{B_n\}$ then converges strongly to

$$B = (M - m)A + mI ,$$

which is obviously self-adjoint. The proof for this case is complete.

The proof for the case when $\{B_n\}$ is nonincreasing is similar, the only difference being that the operators A_{mn} are defined for $n > m$, instead of for $m > n$.

4-14 Since a Hilbert space H is a Banach space, the notions of $L(H,H)$, H^*, etc., are all well defined. We can define an inner product on H^* as follows: For any F and G in H^*, let

$$\langle F, G \rangle = (\overline{f,g}) \tag{15}$$

where the elements f and g in H are such that

$$F(x) = (x,f) \text{ and } G(x) = (x,g) \quad \text{for every } x \in H .$$

The existence and uniqueness of f and g are guaranteed by Theorem 4-3. With this definition of inner product on H^*, we can establish the following theorem:

4-15 *Theorem.* Every Hilbert space is reflexive.

Proof. Let H be a Hilbert space with dual space H^*, and let (15) define the inner product on H^*. We must show that the natural map $x \to \Lambda_x$, where

$$\Lambda_x(F) = F(x) \quad \text{for every } x \in H ,$$

maps H *onto* H^*. That is, given $\mathcal{F} \in H^{**}$, there exists an $x \in H$ such that $x \to \mathcal{F}$ under the natural map. In effect, we must show that given $\mathcal{F} \in H^{**}$, there exists an $x \in H$ such that $\mathcal{F} = \Lambda_x$, or in other words, such that

$$\mathcal{F}(F) = \Lambda_x(F) \quad \text{for all } F \in H^* .$$

Now if $\mathcal{F} \in H^{**}$, then the Riesz representation theorem (4-3) applies to $\mathcal{F}$: there exists a unique $G \in H^*$ such that

$$\mathcal{F}(F) = \langle F,G \rangle \quad \text{for every } F \in H^*$$

But by (15), and a basic property of inner products, we have

$$\langle F,G \rangle = (\overline{f,g}) = (g,f) .$$

Now by the Riesz theorem, as noted at the beginning of the proof,

$$F(g) = (g,f) .$$

This holds for each $F \in H^*$. Hence

$$\mathcal{F}(F) = F(g) \quad \text{for every } F \in H^*.$$

Since

$$\Lambda_g(F) = F(g) \quad \text{for every } F \in H^*,$$

we have

$$\mathcal{F}(F) = \Lambda_g(F) \quad \text{for all } F \in H^*,$$

proving that the natural map is onto. Hence H is reflexive.

4-16 We now give some conditions sufficient in a Hilbert space for an operator to be asymptotically convergent.

Theorem. Let T be a nonexpanding self-adjoint operator on a Hilbert space H. Then T is asymptotically convergent if and only if (-1) is not an eigenvalue of T.

Proof. (i) By definition of norm, for any $x \in H$ we have

$$\|Tx\| \leq \|T\| \, \|x\| .$$

Since $\|Tx\| \geq 0$ always,

$$\|Tx\|^2 \leq \|T\|^2 \, \|x\|^2,$$

and since $\|T\| \leq 1$,

$$(Tx,Tx) \leq \|T\|^2 (x,x) \leq (x,x) .$$

Now using the self-adjoint property of T, it follows that

$$(T^2x,x) = (Tx,Tx) \text{ for every } x \text{ in } H , \tag{16}$$

and therefore

$$(T^2x,x) \leq (x,x) = (Ix,x)$$

where I is the identity operator on H. Hence

$$T^2 \leq I .$$

The relation (16) shows also that

$$T^2 \geq 0 .$$

Now

$$0 \leq T^4 = T^2(T^2) \leq T^2 \leq I \ .$$

In fact, for any $n \geq 1$, we have

$$0 \leq T^{2(n+1)} \leq T^{2n} \leq I \ .$$

Hence $(T^2)^n$ is a bounded monotonic (decreasing) sequence of self-adjoint operators. By Theorem 4-13, there exists a self-adjoint operator Q such that $\{T^{2n}\}$ converges strongly to Q. Since (-1) is not an eigenvalue of T, Theorem 3-16 shows that T is asymptotically convergent.

(ii) Let T be asymptotically convergent and suppose (-1) is an eigenvalue of T. Then if x is an eigenvalue corresponding to (-1), we have

$$T^{2k}\, x = x \quad \text{and} \quad T^{2k+1}\, x = -x \quad \text{for every } k \geq 1 \ .$$

Hence there does not exist an operator Q such that

$$\|T^n x - Qx\| \to 0 \quad \text{as } n \to \infty$$

for this particular x, and therefore T does not converge asymptotically on the space. Since this contradicts the hypothesis, (-1) cannot be an eigenvalue of T. By (i) and (ii), the theorem is proved.

4-17 Theorem. If T is an operator on a Hilbert space H and is of the form

$$T = ASA^{-1} ,$$

where S is a self-adjoint operator on H, and A and A^{-1} are bounded linear operators on H, and if T is asymptotically bounded, then T is asymptotically convergent if and only if (-1) is not an eigenvalue of T.

Proof. (i) We have already seen in part (ii) of the proof of Theorem 4-16 that for T to be asymptotically convergent it is necessary that (-1) not be an eigenvalue of T.

(ii) Suppose now that (-1) is not an eigenvalue of T. First, we see that S is asymptotically bounded, since

$$T^k = (ASA^{-1})^k = AS^kA^{-1} ,$$

and

$$\|S^k\| = \|A^{-1}T^kA\| \leq \|A^{-1}\| \|T^k\| \|A\| .$$

Now S is self-adjoint, and hence (using 4),

$$\|S^2\| = \|S^*S\| = \|S\|^2 .$$

Also, since S^*S is self-adjoint, we have

$$\|S^4\| = \|(S^*S)(S^*S)\| = \|(S^*S)\|^2 = \|S\|^4 .$$

In fact, for any $n \geq 1$, we have

$$\|S^{2n}\| = \|S\|^{2n} .$$

Therefore, since S is asymptotically bounded, there exists a real number M such that

$$\|S\|^{2n} = \|S^{2n}\| \leq M \quad \text{for all } n \geq 1 .$$

Hence we must have

$$\|S\| \leq 1 . \tag{17}$$

Thus S is nonexpanding.

Lemma. If λ is an eigenvalue of S, λ is an eigenvalue of T.

Proof of Lemma. Suppose there exist a scalar λ and an element $y \in H$, with $y \neq 0$, such that

$$Sy = \lambda y .$$

Let $x = Ay$. Since A^{-1} exists, we know $x \neq 0$, and $y = A^{-1}x$. Then

$$SA^{-1}x = \lambda A^{-1}x \ ,$$

and

$$Tx = ASA^{-1}x = A(\lambda A^{-1}x) = \lambda x \ .$$

Hence λ is an eigenvalue of T.

By the lemma above, we have that (-1) is not an eigenvalue of S. In view of (17), we know from Theorem 4-16 that S is asymptotically convergent. That is, there exists an operator Q (which is bounded and linear by 2-5) such that for every $x \in H$, we have

$$\|S^n x - Qx\| \to 0 \quad \text{as } n \to \infty \ .$$

Now A is bounded. Therefore

$$\|AS^n x - AQx\| \to 0 \quad \text{as } n \to \infty$$

for every $x \in H$. Since $A^{-1}x \in H$ for every $x \in H$, we have

$$\|T^n x - AQA^{-1}x\| = \|AS^n A^{-1}x - AQA^{-1}x\| \to 0$$

as $n \to \infty$ for any $x \in H$. Hence T is asymptotically convergent and the theorem is proved.

The two following theorems are applications of the theorems in Chapter 3 and of the preceding theorems in this chapter.

4-18 *Theorem* (Browder-Petryshyn). Let T be a self-adjoint non-expanding mapping on a Hilbert space H, with (-1) not an eigenvalue of T. The iteration scheme defined for any initial approximation x_o by

$$y_k = Ty_{k-1}^{(2^{k-1})} + U_{2^{k-1}}(f) \ , \quad y_o = x_o \ ,$$

where $U_{2^{k-1}} = (I + T^{(2^{k-2})})U_{2^{k-2}} \ , \quad U_1 = I \ ,$ (18)

converges if and only if the equation

$$u - Tu = f \tag{19}$$

has a solution.

Proof. First, we prove that $y_k = x_{(2^{k-1})}$, where x_k is the kth Picard iterate, defined by

$$x_k = Tx_k + f \ ,$$

with initial approximation x_o. The proof is by induction on k.

(i) $y_1 = Ty_o + U_1 f = Ty_o + f = Tx_o + f = x_1$.

(ii) Suppose $y_k = x_{2^{k-1}}$ for some $k \geq 1$. Then

$$y_{k+1} = T^{2^k} y_k + U_{2^k}(f) = T^{2^k} x_{2^k-1} + U_{2^k}(f) .$$

We now note that

$$x_n = T^n x_o + T^{n-1} f + \cdots + Tf + f, \quad \text{for each } n \geq 1,$$

and

$$U_{2^n} f = T^{2^{n-1}} f + \cdots + Tf + f, \quad \text{for each } n \geq 1 .$$

Then

$$y_{k+1} = T^{2^k}(T^{2^k-1} f + \cdots + Tf + f) + (T^{2^k-1} + \cdots + Tf + f)$$

$$= T^{2^{k+1}-1} f + \cdots T^{2^k} f + T^{2^k-1} + \cdots + Tf + f$$

$$= x_{2^{k+1}-1} .$$

This completes the proof.

Therefore the sequence $\{y_k\}$ of elements of H is a subsequence of the sequence $\{x_k\}$ of Picard iterates. Since T is self-adjoint and non-expanding, Theorem 4-16 shows that T is asymptotically convergent. Then by Theorem 3-3, part (a), if a solution of Equation (19) exists, then the Picard iteration sequence will converge to this solution. Hence any subsequence of these iterates will do the same. By part (b) of

Theorem 3-3, if the sequence $\{y_k\}$ converges to an element y of H, then y is a solution of (19). The theorem is proved.

4-19 Let T be a bounded linear operator on a Hilbert space H, and let

$$N(T) = \{x : x \in H, Tx = 0\} .$$

Now N(T) is a closed subspace. It is clear that N(T) is a subspace. To show N(T) is closed, let $\{x_n\}$ be a sequence of elements in H such that

$$x_n \xrightarrow{s} x \quad \text{for some } x \in H .$$

Then, since T is bounded, T is continuous, and thus

$$Tx_n \xrightarrow{s} Tx .$$

But $Tx_n = 0$ for each n. Hence $Tx = 0$, and $x \in N(T)$. Then by Theorem 4-6, there exists a unique pair of linear mappings P and Q such that P maps H into N(T), Q maps H into $(N(T))^{\perp}$, and

$$x = Px + Qx \quad \text{for each } x \in H . \tag{20}$$

Now from Theorem 4-8, part (c), we have

$$(N(T))^{\perp} = \overline{R(T^*)} \ . \tag{21}$$

In view of (21) and (20), we see that

$$PT^*x = TPx = 0 \ ,$$

$$QT^*x = T^*x \ , \text{ and}$$

$$TQx = Tx \quad \text{for every } x \in H \ . \tag{22}$$

4-20 Theorem. Let H be a Hilbert space, T a bounded linear operator in H with $\|T\| > 0$, α a real parameter such that

$$0 < \alpha < 2/\ \|T^*T\| \ ,$$

and let Q denote the projection operator which maps H into $\overline{R(T^*)} = (N(T))^{\perp}$. For any given x_o in H, define the sequence $\{u_k\}$ by the process

$$u_o = x_o, \quad u_k = T^{(2^{k-1})}\, u_{k-1} + U_{2^{k-1}}\, f$$

where

$$f = -\alpha\, T^*g, \quad S = I - \alpha\, T^*T \ ,$$

and U_{2^k} is defined in (18). Then $\{u_k\}$ converges to a solution of

$$Tx = Qg \tag{23}$$

if and only if the latter equation has a solution.

<u>Proof</u>. The operator S is self-adjoint, since for all $x,y \in H$,

$$\begin{aligned}(Sx,y) &= ((I - \alpha T^*T)x,y) = (x,y) - \alpha(T^*Tx,y) \\ &= (x,y) - (x,\alpha(T^*T)^*y) = (x,y - \alpha(T^*T)y) \\ &= (x,(I - \alpha T^*T)y) = (x,Sy) .\end{aligned}$$

To show that S is non-expanding, note that for any $y \in H$,

$$\begin{aligned}\|Sy\|^2 &= (y - \alpha T^*Ty,\ y - \alpha T^*Ty) \\ &= \|y\|^2 - 2\alpha(T^*Ty,y) + \alpha^2 \|T^*Ty\|^2 .\end{aligned} \tag{24}$$

Hence, by the definition of norm and by 4-8, we have

$$\frac{\|T^*Ty\|^2}{(T^*Ty,y)} = \left\{\frac{\|T^*(Ty)\|}{\|Ty\|}\right\}^2 \leq \|T^*\|^2 = \|T^*T\|$$

for $y \neq 0$. Therefore if $y \neq 0$,

$$\frac{2(T^*Ty,y)}{\|T^*Ty\|^2} \geq \frac{2}{\|T^*T\|} > \alpha > 0 ,$$

and

$$2\alpha(T^*Ty,y) > \alpha^2\|T^*Ty\|^2 \ .$$

Then, using (24), it follows that

$$\|Sy\|^2 < \|y\|^2 \ , \quad \text{and} \quad \frac{\|Sy\|}{\|y\|} < 1$$

for $y \neq 0$. Therefore we have

$$\|S\| \leq 1 \ ,$$

and S is non-expanding.

S cannot have -1 as an eigenvalue, since this would imply that for some $x \in H$

$$Sx = x - \alpha T^*Tx = -x \ ,$$

or

$$T^*Tx = \frac{2}{\alpha} x \ .$$

Then we would have

$$\|T^*Tx\| = \frac{2}{\alpha} \|x\| > \|T^*T\| \|x\| \ ,$$

which is contrary to the definition of norm.

Combining the results of the proof up to this point, we see that Theorem 4-18 applies. That is, the iteration scheme defined in expression (18) will converge if and only if Equation (19) has a solution. Suppose a solution of (23) exists. Then, using the relations (22),

$$T^{*}Tx = T^{*}Qg = T^{*}g \ ,$$

or

$$\alpha T^{*}Tx = -\alpha T^{*}g = f \ .$$

Then

$$(-x) - (I - \alpha T^{*}T)(-x) = f, \quad \text{or} \quad u - Tu = f,$$

where $u = -x$, and a solution of (19) exists. By Theorem 4-18, the iterative method defined does converge (to the solution of (19)). Conversely, if the iterative method converges, a solution of (19) exists. That is, for some $u \varepsilon H$ we have

$$u - (I - \alpha T^{*}T)u = - \alpha T^{*}g \ .$$

Then

$$T^{*}Tu = -T^{*}g, \quad \text{and} \quad T^{*}(Tu + g) = 0 \ .$$

Hence $(Tu + g) \varepsilon N(T^{*})$, and therefore

$$Q(Tu + g) = 0, \quad \text{or} \quad QTu = -Qg \ .$$

By applying the relations (22) again, we obtain

$$Tu = QTu = -Qg \ .$$

Hence

$$T(-u) = Qg \ ,$$

and Equation (23) has a solution, and the theorem is proved.

4-21 The last theorem, Theorem 4-20 , is essentially the same as a result of Bialy [4], the only difference being that the scheme of Theorem 4-18 was referred to in Theorem 4-20, whereas Bialy uses the whole Picard sequence. The method of proof used by Bialy, however, relies heavily on spectral representation of the operator T, and thus is less direct than the Browder-Petryshyn method of proof. The proof of 4-20 is not actually given in Browder-Petryshyn [3], but is merely indicated there. The detailed argument we have given requires, in fact, considerable care.

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